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On Common Fixed Point For Compatible mappings in Menger Spaces

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Abstract— In this paper the concept of compatible map in menger space has been applied to prove common fixed point theorem. A fixed point theorem for self maps has been established using the concept of compatibility of pair of self maps.

Index Terms— Common fixed point, menger space, compatible maps, weakly compatible maps.

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1 Introduction

In 1942 Menger [1] has introduced the theory of probabilistic metric spaces in which a distribution function was used instead of non-negative real number as value of the metric. In 1966, Sehgal [2] initiated the study of contraction mapping theorems in probabilistic metric spaces. Since then several generalizations of fixed point Sehgal and Bharucha-Reid [3], Sherwood [4], and Istratescu and Roventa [5] have obtained several theorems in probabilistic metric space. The study of fixed point theorems in probabilistic metric spaces is useful in the study of existence of solutions of operator equations in probabilistic metric space and probabilistic functional analysis. In 2008, Altun and Turkoglu [3] proved two common fixed point theorems on complete PM-space with an implicit relation.

The development of fixed point theory in probabilistic metric spaces was due to Schweizer and Sklar [7] played major role in development of fixed point theory in probabilistic metric spaces. Singh et al. [8] introduced the concept of weakly commuting mappings in probabilistic metric spaces. The concept of weakly-compatible mappings is most general as every commuting pair is R-weakly commuting, each pair of R-weakly commuting mappings is compatible and each pair of compatible mappings is weakly compatible but the converse is not true. Kumar and Chugh [9] established some common fixed point theorems in metric spaces by using the ideas of pointwise R-weak commutativity and reciprocal continuity of mappings. A fixed point theorem concerning probabilistic contractions satisfying an implicit relation was proved by Mihet [10] in 2005.

The main object of this paper is to obtain fixed point theorems in the setting of Menger space using concept of compatibility.

2 Preliminaries.

we recall some definitions and known results.

Definition 2.1. [11] A mapping $F : R \to R^+$ is called a *distribution* if it is non-decreasing left continuous with inf $\{F(t) : t \in R\} = 0$ and sup $\{F(t) : t \in R\} = 1$. .

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, t \le 0 \\ 1, t > 0 \end{cases}$$

Definition 2.2. [11] A mapping t : [0,1] x [0,1] → [0,1] is called a continuous *t-norm* if it satisfies the following conditions:

(t-1) t is commutative and associative;
(t-2) $t(a,1) = a$ for all $a \in [0,1]$;
(t-3) $t(a,b) \le t(c,d)$ for $a \le c$, $b \le d$.

The following are the basic t-norms:

$T_M(x,y) = \mathrm{Min}\{x.y\}$
$T_P(x,y) = x \cdot y$
$T_L(x,y) = \mathrm{Max}\{x+y-1, 0\}$.

Each t-norm T can be extended [14] (by associativity) in a unique way taking for $(x_1, x_2, \ldots x_n) \in [0,1]^n$, $(n \in N)$ the values $T^1(x_1, x_2) = T(x_1, x_2)$ and $T^n(x_1, x_2 \ldots x_{n+1}) = T(T^{n-1}(x_1, x_2 \ldots x_n), x_{n+1})$ for $n \ge 2$ and $x_i \in [0,1]$, for all $i \in \{1, 2, \ldots n+1\}$.

Definition 2.3. [11] A probabilistic metric space (PM-space) is an ordered pair (X,F) consisting of a non empty set X and a function F: X×X →L, where L is the collection of all distribution functions and the value of F at $(u,v) \in X \times X$ is represented by $F_{u,v}$. The function $F_{u,v}$ is assumed to satisfy the following conditions:

(PM – 1) $F_{u,v}(x) = 1$, for all $x > 0$ if and only if $u = v$;
(PM – 2) $F_{u,v}(0) = 0$;
(PM – 3) $F_{u,v} = F_{v,u}$;
(PM – 4) If $F_{u,v}(x) = 1$ and $F_{v,w}(x) = 1$ then

$F_{u,w}(x+y) = 1$ for all u,v,w in X and x,y > 0 .

Definition 2.4. [11] A *Menger space* is a triplet (X, F, t) where (X,F) is a PM-space and t is a t-norm such that the inequality

(PM – 5) $F_{u,w}(x+y) \geq t\{ F_{u,v}(x) , F_{v,w}(x) \}$ for all u,v,w in X and x,y > 0 .

Definition 2.5. [11] A sequence $\{x_n\}$ in a Menger space (X, F, t) is said to *converges to a point* x in X if and only if for each $\varepsilon > 0$ and t > 0, there is an integer $M(\varepsilon) \in N$ such that

$$F_{x_n,x}(\varepsilon) > 1 - t \text{ for all } n \geq M(\varepsilon)$$

Definition 2.6. [11] The sequence $\{x_n\}$ is said to be *Cauchy sequence* if for $\varepsilon > 0$ and t > 0, there is an integer $M(\varepsilon) \in N$ such that

$$F_{x_n,x_m}(\varepsilon) > 1 - t \text{ for all } n,m \geq M(\varepsilon)$$

Definition 2.7. [11] A Menger PM-space (X, F, t) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

A complete metric space can be treated as a complete Menger space in the following way:

Lemma 2.1 [11] If (X,d) is a metric space then the metric d induces mappings F: X × X → L , defined by $F_{p,q} = H(x-d(p,q))$, p, q ∈ X, where H(k) = 0 for k ≤ 0 and H(k) = 1 for k > 0.

Further if, t : [0,1] x [0,1] → [0,1] is defined by t(a,b) = min{a,b}. Then (X, F, t) is a Menger space. It is complete if (X,d) is complete.

The space (X, F, t) so obtained is called the induced Menger space.

Definition 2.8. [12] Self mappings A and S of a Menger space (X, F, t) are said to be *compatible* if

$$F_{ASx_n,SAx_n}(x) \to u \text{ for all } x > 0,$$

whenever $\{x_n\}$ is a sequence in X such that $ASx_n , SAx_n \to u$ for some u in X, as $n \to \infty$.

Definition 2.9 [16]. Two maps A and B are said to be weakly compatible if they commute at a coincidence point.

Lemma 2.2 [12] If S and T are compatible self maps of a Menger space (X, F, t) where t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$ and $Sx_n , Tx_n \to u$ for some u in X. Then $TSx_n \to u$ provided S is continuous.

Lemma 2.3 Self-mappings A and B of a Menger space (X, F, t) are compatible, then they are weak compatible.

The converse is not true as seen in following example.

Example 2.1 Let X = [0,2] with usual metric d where d(x,y) = | x – y | for all x and y in X.

Let $F_{x,y} = \dfrac{t}{t + d(x,y)}$ for all x and y in X and t > 0.

Define: $A(x) = \begin{cases} x; x \in [0,1) \\ 2; x \in [1,2] \end{cases}$ and

$$S(x) = \begin{cases} 2 - x; x \in [0,1) \\ 2; x \in [1,2] \end{cases}$$

Let $x_n = 1 - \frac{1}{n}$ then $Ax_n = 1 - \frac{1}{n}$ and $Sx_n = 1 + \frac{1}{n}$

Thus $Ax_n \to 1$ and $Sx_n \to 1$ and hence x = 1.

Also $ASx_n = 2$ and $SAx_n = 1 + \frac{1}{n}$.

Now $\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = \lim_{n\to\infty} F_{2,1+\frac{1}{n}}(t) = \frac{t}{t+1} < 1$

for all t > 0.

Hence A and S are not compatible.

Again $\lim_{n\to\infty} F_{ASx_n,Sx}(t) = \lim_{n\to\infty} F_{2,2}(t) = 1$

Hence A and S are semi compatible and $\lim_{n\to\infty} F_{SAx_n,Ax}(t) = \lim_{n\to\infty} F_{1+\frac{1}{n},2}(t) = \frac{t}{t+1} < 1$ for t > 0.

Therefore it is clear that S, A are not semi compatible.

Now we will show that the semi compatible pair (A, S) is also weakly compatible .

Now coincidence points of A and S are in [1, 2].

Therefore for any x in [1, 2], we have

Ax = Sx = 2 and AS(x) = 2 = SA(x) and A(2) = 2 = S(2)

Thus (A, S) is weakly compatible.

3 MAIN RESULT

Theorem 1. *Theorems, Let* (X,F,t) *be a complete Menger space with continuous* t*-norm* t *and let* $h: X \to X$, $k: X \to X$, $f: X \to h(X)$ *and* $g: X \to k(X)$ *be continuous mapping such that* (f,k) *and* (g,h) *are compatible pairs. Further, suppose that for all* $x, y \in X$ *and for all* $\varepsilon > 0$ *the following inequality holds*

$$F_{fx,gy}(\varepsilon) \geq F_{kx,hy}(\phi(\varepsilon))$$

Where $\phi: R^+ \to R^+$ *is an increasing function such that* $\lim_{n\to\infty} \phi^n(t) = \infty$ *for all* $t > 0$. *If the sequence* $\{y_n\}_{n\in N}$ *formed by*

$$y_{2n-1} = gx_{2n-1} = kx_{2n},$$
$$y_{2n} = fx_{2n} = hx_{2n+1},\ n \in N$$

is probabilistically bounded for some $x_1 \in X$, *then there exists a unique common fixed point for the mappings* f, g, h *and* k .

Proof. Let $\{y_n\}$ be the sequence satisfying the given condition.

We shall show that $\{y_n\}_{n\in N}$ is a Cauchy sequence.

For that, we shall show that

$$\lim_{m,p\to\infty} F_{y_m,y_p}(\varepsilon) = H(\varepsilon), \text{ for every } \varepsilon \in R .$$

If $m = 2i$ and $p = 2j - 1$ (let $j > i$) then we have

$$F_{y_{2i},y_{2j-1}}(\varepsilon) = F_{fx_{2i},gx_{2j-1}}(\varepsilon) \geq F_{kx_{2i},hx_{2j-1}}(\phi(\varepsilon))$$
$$= F_{fx_{2j-2},gx_{2i-1}}(\phi(\varepsilon)) \geq F_{kx_{2j-2},hx_{2i-1}}(\phi^2(\varepsilon))$$
$$= F_{fx_{2i-2},gx_{2j-1}}(\phi^2(\varepsilon)) \geq F_{fx_0,gx_{2j-1-2i}}(\phi^{2i}(\varepsilon))$$
$$\geq \sup_{t<\phi^{2i}(\varepsilon)} \inf_{n,k\in N} F_{y_n,y_k}(t) = D_{\{y_n\}_{n=1}^{\infty}}(\phi^{2i}(\varepsilon)).$$

Since $\{y_n\}_{n\in N}$ is probabilistically bounded, by considering $i \to \infty$ and $j \to \infty$, we get

$$\lim_{i\to\infty} D_{\{y_n\}_{n=1}^{\infty}}(\phi^{2i}(\varepsilon)) = (\varepsilon).$$

By repeating this process, we can prove a similar result for $m = 2i-1$ and $p = 2j$.

If m and p are both even or both odd, we proceed as follows.

$$F_{y_{2i},y_{2j}}(\varepsilon) \geq t(F_{y_{2i},y_{2i+1}}(\tfrac{\varepsilon}{2}), F_{y_{2i+1},y_{2j}}(\tfrac{\varepsilon}{2})) \to$$
$$t(H(\varepsilon), H(\varepsilon)) = H(\varepsilon).$$
$$F_{y_{2i-1},y_{2j-1}}(\varepsilon) \geq t(F_{y_{2i-1},y_{2i}}(\tfrac{\varepsilon}{2}), F_{y_{2i},y_{2j-1}}(\tfrac{\varepsilon}{2})) \to$$
$$t(H(\varepsilon), H(\varepsilon)) = H(\varepsilon).$$

If $i \to \infty$ and $j \to \infty$, for all $\varepsilon > 0$.

Thus we have proved that is $\{y_n\}_{n\in N}$ a Cauchy sequence in X which means that there exists $y^* \in X$ such that $\lim_{n\to\infty} y_n = y^*$.

To prove that $fy^* = gy^* = hy^* = ky^*$, we proceed as follows.

$$fy^* = f\lim_{n\to\infty} kx_{2n} = \lim_{n\to\infty} fkx_{2n} = \lim_{n\to\infty} kfx_{2n} = k\lim_{n\to\infty} fx_{2n} = ky^*.$$

$$gy^* = g\lim_{n\to\infty} hx_{2n+1} = \lim_{n\to\infty} ghx_{2n+1} = \lim_{n\to\infty} hgx_{2n+1} = h\lim_{n\to\infty} gx_{2n+1} = hy^*.$$

Since $F_{fy^*,gy^*}(\varepsilon) \geq F_{ky^*,hy^*}(\phi(\varepsilon)) = F_{fy^*,gy^*}(\phi(\varepsilon)) \geq \ldots\ldots \geq F_{fy^*,gy^*}(\phi^n(\varepsilon)) \to H(\varepsilon)$. For all $\varepsilon > 0$.

Thus $fy^* = gy^* = hy^* = ky^*$.

The point fy^* is a fixed point for the mapping f, g, h and k. We shall show this for the mapping f. The proof for the mappings g, h and k is similar.

From compatibility of (f,k) we obtain that

$$kfy^* = kf\lim_{n\to\infty} y_n = \lim_{n\to\infty} kfy_n = \lim_{n\to\infty} fky_n = fk\lim_{n\to\infty} y_n = fky^* = ffy^*.$$

Further,

$$F_{ffy^*,fy^*}(\varepsilon) = F_{ffy^*,gy^*}(\varepsilon) \geq F_{kfy^*,hy^*}(\phi(\varepsilon)) = F_{ffy^*,fy^*}(\phi(\varepsilon)) \geq \ldots \geq F_{ffy^*,fy^*}(\phi^n(\varepsilon)) \to H(\varepsilon)$$

for n $\to \infty$ for $\varepsilon > 0$, which means that fy^* is a common fixed point for the mappings f, g, h and k.

Uniqueness: For uniqueness let if possible we suppose that there exists another common fixed point $z \in Z$, therefore we get,

$$F_{fy^*,z}(\varepsilon) = F_{ffy^*,gz}(\varepsilon) \geq F_{kfy^*,hz}(\phi(\varepsilon)) = F_{ffy^*,gz}(\phi(\varepsilon)) \geq \ldots\ldots \geq F_{ffy^*,z}(\phi^n(\varepsilon)) \to H(\varepsilon)$$

for $n \to \infty$ for $\varepsilon > 0$, which means that fy^* is a unique common fixed point for the mappings f, g, h and k.

Hence the theorem. □

4 CONCLUSION

In this paper, we have described common fixed point theorems for four mappings in Menger space by compatibility. This idea can be implemented in the other metric spaces.

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REFERENCES

[1] K. Menger, Statistical metrics, Proc. Nat. Acad. Sci. U.S.A. 28 (1942), 535–537.
[2] V.M. Sehgal, Some fixed point theorems in function analysis and probability, Ph.D dissertation, Wayne State Univ. Michigan (1966).
[3] V.M. Sehgal and A.T. Bharucha-Reid, Fixed points of contraction mappings on probabilistic metric spaces, Math. Systems Theory 6 (1972), 97–102.
[4] H. Sherwood, Complete probabilistic metric spaces, Z. wahrscheinlichkeits theorie and verw. Grebiete 20 (1971), 117–128.
[5] V.I. Istratescu and I. Sacuiu, Fixed point theorem for contraction mappings on probabilistic metric spaces, Rev. Roumaine Math. Pures. Appl. 18 (1973), 1375–1380.
[6] I. Altun and D. Turkoglu, Some fixed point theorems on fuzzy metric spaces with implicit relations, Commun. Korean Math. Soc., 23(1)(2008), 111-124.
[7] B. Schweizer and A. Sklar, Probabilistic Metric Spaces, North Holland(Amsterdam, 1983).
[8] S.L. Singh, B.D. Pant and R. Talwar, Fixed points of weakly commuting mappings on Menger spaces, Jnanabha 23 (1993),

115–122.

[9] S. Kumar and R. Chugh, Common fixed point theorems using minimal commutativity and reciprocal continuity conditions in metric spaces, Sci. Math. Japan 56 (2002), 269–275.

[10] D. Mihet, A generalization of a contraction principle in probabilistic metric spaces, Part II, Int. J. Math. Math. Sci. 2005 (2005), 729–736.

[11] S.N. Mishra, Common fixed points of compatible mappings in PM-spaces, Math. Japon. 36(2), (1991), 283-289.

[12] A. Jain and B. Singh, Common fixed point theorem in Menger space through compatible maps of type (A), Chh. J. Sci. Tech. 2 (2005), 1-12.

[13] B. Singh and S. Jain, Semi-compatibility and fixed point theorem in Menger space, Journal of the Chungcheong Mathematical Society 17 (1), (2004), 1-17.

[14] E. P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Trend in Logic 8, Dordrecht (2000).

[15] R. A. Rashwan & A. Hedar: On common fixed point theorems of compatible mappings in Menger spaces. Demonstratio Math. 31 (1998), no. 3, 537–546.

[16] G. Jungck & B. E. Rhoades: Fixed points for set valued functions without continuity. Indian J. Pure Appl. Math. 29 (1998), no. 3, 227–238.